

On the generalization of quantum state comparison

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We investigate the unambiguous comparison of quantum states in a scenario that is more general than the one that was originally suggested by Barnett *et al.* First, we find the optimal solution for the comparison of two states taken from a set of two pure states with arbitrary *a priori* probabilities. We show that the optimal coherent measurement is always superior to the optimal incoherent measurement. Second, we develop a strategy for the comparison of two states from a set of N pure states, and find an optimal solution for some parameter range when $N = 3$. In both cases we use the reduction method for the corresponding problem of mixed state discrimination, as introduced by Raynal *et al.*, which reduces the problem to the discrimination of two pure states only for $N = 2$. Finally, we provide a necessary and sufficient condition for unambiguous comparison of mixed states to be possible.

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I. INTRODUCTION

The laws of quantum mechanics do not allow the perfect discrimination of two non-orthogonal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$. Consequently, given a set of non-orthogonal states $\{|\psi_1\rangle, |\psi_2\rangle\}$, it is also impossible to find out with probability one whether two quantum states, drawn from this set, are identical (namely, the total state is either $|\psi_1\psi_1\rangle$ or $|\psi_2\psi_2\rangle$) or different (i.e. the total state is either $|\psi_1\psi_2\rangle$ or $|\psi_2\psi_1\rangle$). What is the optimal probability of success, when no errors are allowed? This problem has been introduced by Barnett, Chefles and Jex [1] and is called unambiguous quantum state comparison. It has been solved for the case that the *a priori* probabilities for the two ensemble states are equal [1]. The task of determining whether C given states taken from a set of N pure states with equal *a priori* probabilities are identical or not has been investigated in [2, 3].

In this article, we consider the most general case of unambiguous state comparison, also admitting mixed states. We provide sufficient and necessary conditions, for which this task can succeed. Furthermore, the comparison of two states drawn from a set of N pure states with arbitrary *a priori* probabilities is investigated, and an optimal solution is found for the case $N = 2$, as well as for a range of parameters in the case $N = 3$, using the reduction techniques for mixed state discrimination developed in [4]. This method is also applied for general N . We address the question of how much can be gained in the optimal coherent strategy (i.e. with global measurements on the two given states), as compared to the best incoherent strategy (i.e. consecutive measurements).

Our paper is organized as follows: in section II, we define the most general state comparison problem, and explain the connection to mixed state discrimination. In section III, we find the optimal solution for comparing

two states, drawn from a set of two states. In section IV, we develop the formalism for the comparison of two out of N states, and apply it to $N = 3$. In section V we derive sufficient and necessary conditions for the general task of mixed state comparison to be successful, before concluding in section VI.

II. GENERAL STATE COMPARISON

Let us define the task of state comparison in the most general way: *Given C quantum states of arbitrary dimension, each of them taken from a set of N possible (in general mixed) quantum states $\{\pi_1, \dots, \pi_N\}$ that occur with corresponding a priori probabilities $\{q_1, \dots, q_N\}$. Unambiguous state comparison “ C out of N ” is performed by doing a measurement, which allows with probability P to decide without doubt whether all C states are equal, or whether at least one of them is different. The best possible probability of success P_{opt} is reached in optimal state comparison.*

A measurement is most generally described as a positive operator-valued measurement (POVM), i.e. a decomposition of the identity operator into a set of n positive operators [5]

$$F_1, \dots, F_n \geq 0, \quad \text{satisfying} \quad \sum_i F_i = \mathbb{1}. \quad (1)$$

The probability for a system in a state ϱ_k to yield the outcome corresponding to F_i is given by $p_k \text{tr}(F_i \varrho_k)$, where p_k is the *a priori* probability for the system being in state ϱ_k . For the task of unambiguous state comparison, we need at least two measurements F_a and F_b , having vanishing probabilities in the case where the total state is composed of different or equal states, respectively. This means, that for all $(p_k, \varrho_k) \in \{(q_{i_1} \dots q_{i_C}, \pi_{i_1} \otimes \dots \otimes \pi_{i_C}) \mid i_1, \dots, i_C \in \{1, \dots, N\}\}$ we demand

$$p_k \text{tr}(F_a \varrho_k) > 0 \Leftrightarrow \exists m: \varrho_k = \pi_m^{\otimes C}, \quad (2a)$$

$$p_k \text{tr}(F_b \varrho_k) > 0 \Leftrightarrow \nexists m: \varrho_k = \pi_m^{\otimes C}. \quad (2b)$$

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However, measurements which satisfy this defining property will in general not sum up to the identity, thus admitting the inconclusive measurement $F_\gamma = \mathbb{1} - F_a - F_b$, which has to be a positive operator. In order to find an *optimal* solution to the problem, one has to minimize the probability for the inconclusive answer $\sum_k p_k \text{tr}(F_\gamma \varrho_k)$, or equivalently maximize the rate of success given by

$$P = \sum_k p_k \text{tr}((F_a + F_b) \varrho_k). \quad (3)$$

The problem of finding the optimal measurement for state comparison can be addressed by considering the optimal solution of a related problem, namely unambiguous state discrimination. Here, two states ρ_a and ρ_b have to be distinguished without error, but admitting an inconclusive answer. In order to see the connection between the two tasks, consider the mixed states

$$\rho_a = \frac{1}{\eta_a} \sum_i (q_i \pi_i)^{\otimes C}, \quad (4a)$$

$$\rho_b = \frac{1}{\eta_b} \left(\sum_i q_i \pi_i \right)^{\otimes C} - \frac{\eta_a}{\eta_b} \rho_a, \quad (4b)$$

with *a priori* probabilities

$$\eta_a = \sum_i q_i^C \quad \text{and} \quad \eta_b = 1 - \eta_a. \quad (4c)$$

Now, a POVM, which satisfies (2) also has

$$F_a \rho_b = 0 \quad \text{and} \quad F_b \rho_a = 0, \quad (5)$$

and furthermore the probability of success (3) which has to be optimized can be rewritten as

$$P = \eta_a \text{tr}(F_a \rho_a) + \eta_b \text{tr}(F_b \rho_b). \quad (6)$$

These equations are characteristic for unambiguous state discrimination. Thus an optimal solution to the problem of unambiguous discrimination (UD) of ρ_a and ρ_b , which in addition satisfies (2), is also the optimal solution to the related problem of unambiguous state comparison. The task of optimal UD of mixed states has been studied in the literature [4, 6, 7, 8, 9].

III. STATE COMPARISON “TWO OUT OF TWO”

We first consider explicitly the most simple case of state comparison, namely “two out of two” with the states subject to comparison being pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, both of which are vectors in a Hilbert space of any dimension. The two states may appear with arbitrary (but non-vanishing) *a priori* probabilities q_1 and q_2 . The trivial cases, where both states are co-linear or orthogonal are not considered. Without loss of generality the phase between the two states can be chosen to be real,

so that their overlap is determined by their relative angle ϑ ,

$$\cos \vartheta := \langle \psi_1 | \psi_2 \rangle \in]0, 1[. \quad (7)$$

We consider the related UD problem of the corresponding mixed states, which are according to the equations (4) given by

$$\rho_a = \frac{1}{\eta_a} (q_1^2 |\psi_1 \psi_1\rangle\langle \psi_1 \psi_1| + q_2^2 |\psi_2 \psi_2\rangle\langle \psi_2 \psi_2|), \quad (8a)$$

$$\rho_b = \frac{1}{2} (|\psi_1 \psi_2\rangle\langle \psi_1 \psi_2| + |\psi_2 \psi_1\rangle\langle \psi_2 \psi_1|), \quad (8b)$$

appearing with *a priori* probabilities

$$\eta_a = q_1^2 + q_2^2 \quad \text{and} \quad \eta_b = 2q_1 q_2. \quad (8c)$$

Note, that $\eta_a \geq \eta_b$ always holds. In what follows, we construct an optimal solution of this related UD problem and then show that the POVM of this solution satisfies (2), thus providing an optimal solution of the unambiguous state comparison task.

A. Reduction to the Non-Trivial Subspace

It has been shown by Raynal, Lütkenhaus and van Enk [4] that the optimal UD of mixed states can be reduced to a subspace of the Hilbert space in such a way, that the relevant density matrices, acting on the reduced space, have equal rank and their kernels form non-orthogonal subspaces, the intersection of which is zero. This is achieved in two reduction steps: In the *first reduction step*, the Hilbert space is reduced to its non-trivial part, removing that part of the Hilbert space, where no UD is possible at all. We will denote this reduced space as \mathcal{H} . It is given by the particular space, where

$$\mathcal{S}_{\rho_a} \cap \mathcal{S}_{\rho_b} = 0 \quad \text{and} \quad \mathcal{K}_{\rho_a} \cap \mathcal{K}_{\rho_b} = 0 \quad (9)$$

holds. Here \mathcal{K}_ρ is the kernel of ρ and \mathcal{S}_ρ its support, defined as the ortho-complement to the kernel [12]. Thus, \mathcal{H} contains only the direct sum of the support of ρ_a and ρ_b , i.e. $\mathcal{H} = \mathcal{S}_{\rho_a} \oplus \mathcal{S}_{\rho_b}$.

For our system, we have

$$\mathcal{S}_{\rho_a} = \text{span}(|\psi_1 \psi_1\rangle, |\psi_2 \psi_2\rangle), \quad (10a)$$

$$\mathcal{S}_{\rho_b} = \text{span}(|\psi_1 \psi_2\rangle, |\psi_2 \psi_1\rangle), \quad (10b)$$

which already satisfy $\mathcal{S}_{\rho_a} \cap \mathcal{S}_{\rho_b} = \{0\}$ due to the linear independence of $|\psi_1\rangle$ and $|\psi_2\rangle$. For the further calculation it is convenient to rewrite both supports in an appropriate basis of \mathcal{H} . Therefore consider complementary normalized vectors $|\bar{\psi}_1\rangle, |\bar{\psi}_2\rangle \in \text{span}(|\psi_1\rangle, |\psi_2\rangle)$, which are in the same plane as $|\psi_1\rangle$ and $|\psi_2\rangle$, but orthogonal to the corresponding vector, i.e. $|\bar{\psi}_1\rangle \perp |\psi_1\rangle$ and $|\bar{\psi}_2\rangle \perp |\psi_2\rangle$. Then, an orthonormal basis of \mathcal{H} is given by

$$|e_{1,2}\rangle = \frac{1}{\sqrt{2}n_{\pm}} (|\psi_1 \psi_1\rangle \pm |\psi_2 \psi_2\rangle), \quad (11a)$$

$$|e_{3,4}\rangle = \frac{1}{\sqrt{2}n_{\pm}} (|\bar{\psi}_1 \bar{\psi}_2\rangle \pm |\bar{\psi}_2 \bar{\psi}_1\rangle), \quad (11b)$$

with $n_{\pm} = \sqrt{1 \pm \cos^2 \vartheta}$. In equation (11a), the $+$ ($-$)-sign refers to the index 1(2) and in (11b) to 3(4) respectively.

By this choice, one immediately has $\mathcal{K}_{\rho_a} = \text{span}(|e_3\rangle, |e_4\rangle)$ and $|e_2\rangle \in \mathcal{K}_{\rho_b}$. Let us denote by $P_+ = |e_1\rangle\langle e_1| + |e_3\rangle\langle e_3|$ ($P_- = |e_2\rangle\langle e_2| + |e_4\rangle\langle e_4|$) the projector onto that subspace, which is symmetric (antisymmetric) under exchanging $|\psi_1\rangle$ and $|\psi_2\rangle$. Then, due to $|\psi_1\psi_2\rangle \in \mathcal{S}_{\rho_b}$,

$$|\gamma\rangle := \frac{\sqrt{2}}{n_+} P_+ |\psi_1\psi_2\rangle = \frac{\sqrt{2}}{n_+} P_+ |\psi_2\psi_1\rangle \in \mathcal{S}_{\rho_b} \quad (12)$$

must hold, where $|\gamma\rangle$ is normalized and has the components

$$|\langle e_1|\gamma\rangle| = \frac{2\cos\vartheta}{n_+^2} \quad \text{and} \quad |\langle e_3|\gamma\rangle| = \frac{\sin^2\vartheta}{n_+^2}. \quad (13)$$

Since $P_- + P_+ = \mathbb{1}_{\mathcal{H}}$, the second spanning vector of \mathcal{S}_{ρ_b} has to be $P_- |\psi_1\psi_2\rangle = -P_- |\psi_2\psi_1\rangle$. This vector, however, cannot have any component in direction of $|e_2\rangle \in \mathcal{K}_{\rho_b}$ and therefore has to be parallel to $|e_4\rangle$. Thus, we finally write the non-trivial Hilbert space \mathcal{H} as

$$\mathcal{H} = \mathcal{S}_{\rho_a} \oplus \mathcal{S}_{\rho_b} \equiv \text{span}(|e_1\rangle, |e_2\rangle) \oplus \text{span}(|\gamma\rangle, |e_4\rangle). \quad (14)$$

Due to the particular choice of basis, we further find $\mathcal{K}_{\rho_b} = \text{span}(|\gamma^\perp\rangle, |e_2\rangle)$, where $|\gamma^\perp\rangle$ is a normalized vector satisfying $|\gamma^\perp\rangle \perp |\gamma\rangle$ and $P_- |\gamma^\perp\rangle = P_- |\gamma\rangle \equiv 0$.

B. Optimal Solution

In the *second reduction step* shown in [4], one reduces the space by those parts, which allow perfect UD. These parts are given by

$$\mathcal{K}_a^\cap := \mathcal{K}_{\rho_a} \cap \mathcal{S}_{\rho_b} \quad \text{and} \quad \mathcal{K}_b^\cap := \mathcal{K}_{\rho_b} \cap \mathcal{S}_{\rho_a}. \quad (15)$$

The Hilbert space \mathcal{H} can then be decomposed into

$$\mathcal{H} = \mathcal{H}' \oplus \mathcal{K}_a^\cap \oplus \mathcal{K}_b^\cap, \quad (16)$$

where \mathcal{H}' is conveniently chosen to be the orthocomplement of $\mathcal{K}_a^\cap \oplus \mathcal{K}_b^\cap$. Denoting by $P_{\mathcal{H}'}$ the projector onto \mathcal{H}' , and further writing ζ_a, ζ_b for appropriate normalization constants, the density matrices

$$\rho'_a = \frac{1}{\zeta_a} P_{\mathcal{H}'} \rho_a P_{\mathcal{H}'} \quad \text{and} \quad \rho'_b = \frac{1}{\zeta_b} P_{\mathcal{H}'} \rho_b P_{\mathcal{H}'} \quad (17)$$

are states acting on \mathcal{H}' and having *a priori* probabilities

$$\eta'_a = \frac{\eta_a \zeta_a}{\zeta} \quad \text{and} \quad \eta'_b = 1 - \eta'_a, \quad (18)$$

where $\zeta := \zeta_a \eta_a + \zeta_b \eta_b$. Suppose that P' is the optimal rate of success for this reduced problem. Then the optimal rate of success of the complete problem was shown [4] to be

$$P_{\text{opt}} = 1 - (1 - P')\zeta. \quad (19)$$

In our basis, we immediately find

$$\mathcal{K}_a^\cap = \text{span}(|e_3\rangle, |e_4\rangle) \cap \mathcal{S}_{\rho_b} = \text{span}(|e_4\rangle), \quad (20a)$$

$$\mathcal{K}_b^\cap = \text{span}(|\gamma^\perp\rangle, |e_2\rangle) \cap \mathcal{S}_{\rho_a} = \text{span}(|e_2\rangle), \quad (20b)$$

since $|\gamma\rangle \nparallel |e_3\rangle$ and $|\gamma^\perp\rangle \nparallel |e_1\rangle$ holds.

Now the optimization problem can be reduced to $\mathcal{H}' = \text{span}(|e_1\rangle, |e_3\rangle)$. Since the remaining problem is two-dimensional, it can be considered as the well-known discrimination of pure states. Indeed, the problem reduces to the UD of

$$\rho'_a = \frac{1}{\zeta_a} P_+ \rho_a P_+ = |e_1\rangle\langle e_1|, \quad (21a)$$

$$\rho'_b = \frac{1}{\zeta_b} P_+ \rho_b P_+ = |\gamma\rangle\langle\gamma|. \quad (21b)$$

Calculating the normalization factors $\zeta_a = \text{tr}(P_+ \rho_a)$ and $\zeta_b = \text{tr}(P_+ \rho_b)$, one obtains $\zeta_a = \zeta_b = \zeta = \frac{1}{2} n_+^2$ and thus the *a priori* probabilities of the reduced problem remain unchanged, $\eta'_a \equiv \eta_a$ and $\eta'_b \equiv \eta_b$. Jaeger and Shimony have derived [10] the optimal UD of two pure states with an unbalanced probability distribution. Using their result for the discrimination between $|e_1\rangle$ and $|\gamma\rangle$, the optimal rate of success for UD of ρ_a and ρ_b calculates to

$$P_{\text{opt}} = \begin{cases} 1 - 2\sqrt{\eta_a \eta_b} \cos \vartheta & \text{if } (*) \\ \frac{n_+^2}{n_+^2} (1 - \frac{\eta_b}{2} \sin^2 \vartheta) & \text{else,} \end{cases} \quad (22)$$

where $(*)$ is the condition

$$\cos \vartheta < \sqrt{\frac{\eta_a}{\eta_b}} \left(1 - \sqrt{\frac{\eta_a - \eta_b}{\eta_a}} \right). \quad (*)$$

Further, the optimal POVM of the reduced problem is given by

$$F'_a = \alpha |\gamma^\perp\rangle\langle\gamma^\perp| \quad \text{and} \quad F'_b = \beta |e_3\rangle\langle e_3|. \quad (23)$$

In the region, where $(*)$ holds,

$$\alpha = \frac{1 - \sqrt{\frac{\eta_b}{\eta_a}} |\langle e_1|\gamma\rangle|}{|\langle e_3|\gamma\rangle|^2}, \quad (24a)$$

$$\beta = \frac{1 - \sqrt{\frac{\eta_a}{\eta_b}} |\langle e_1|\gamma\rangle|}{|\langle e_3|\gamma\rangle|^2}, \quad (24b)$$

and $\alpha = 1, \beta = 0$ elsewhere. The optimal measurement of the full problem is then given by

$$F_a = F'_a + P_{\mathcal{K}_b^\cap} \quad \text{and} \quad F_b = F'_b + P_{\mathcal{K}_a^\cap}, \quad (25)$$

where $P_{\mathcal{K}_b^\cap} \equiv |e_2\rangle\langle e_2|$ and $P_{\mathcal{K}_a^\cap} \equiv |e_4\rangle\langle e_4|$. The fact that the projectors $|e_2\rangle\langle e_2|$ and $|e_4\rangle\langle e_4|$ have to be part of the optimal POVMs F_a and F_b , respectively, was already obvious from the structure of the kernels and supports, since $|e_2\rangle$ and $|e_4\rangle$ are orthogonal and part of either \mathcal{S}_{ρ_a} or \mathcal{S}_{ρ_b} .

Now one easily verifies, that condition (2) holds for this measurement, by noting that $|\langle \psi_1 \psi_1 | e_2 \rangle|^2 =$

$|\langle\psi_2\psi_2|e_2\rangle|^2 > 0$ and $|\langle\psi_1\psi_2|e_4\rangle|^2 = |\langle\psi_2\psi_1|e_4\rangle|^2 > 0$. Thus we have found the optimal solution for unambiguous two-dimensional state comparison. Furthermore, as we discuss in the following, this solution is *always* better than a separable measurement on both states, which becomes manifest by the fact, that F_a and F_b are not separable, i.e. the partial transpose fails to be positive semidefinite.

C. Discussion

In the literature, an optimal solution for the problem of state comparison has only been found for the case of equal probabilities. Barnett, Chefles and Jex [1] showed, that in this case the optimal rate of success is given by $P = 1 - \cos\vartheta$, which is our result for $q_1 = q_2 = \frac{1}{2}$. This particular result was also obtained by Rudolph, Spekkens and Turner [7], by providing a general upper and lower bound for the rate of success of an UD of mixed states. Their upper bound matches our result only in situations, where (*) holds. On the other hand, their lower bound turns out to match our optimal result for all parameters and thus our calculation has proven, that their lower bound is indeed optimal for the UD of ρ_a and ρ_b .

Let us compare our result with the naïve incoherent strategy, where both states are measured consecutively. The straightforward approach of the optimal POVM $\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_?\}$ for unambiguous discrimination between $|\psi_1\rangle$ and $|\psi_2\rangle$, leads to

$$F_a^{\text{sep}} = \tilde{F}_1 \otimes \tilde{F}_1 + \tilde{F}_2 \otimes \tilde{F}_2, \quad (26a)$$

$$F_b^{\text{sep}} = \tilde{F}_1 \otimes \tilde{F}_2 + \tilde{F}_2 \otimes \tilde{F}_1. \quad (26b)$$

This naïve method is indeed the *optimal* separable measurement, as shown in appendix A. It has a rate of success given by the square of the success probability for unambiguous discrimination of $|\psi_1\rangle$ and $|\psi_2\rangle$, i.e. [10]

$$P_{\text{sep}} = \begin{cases} (1 - 2\sqrt{q_1 q_2} \cos\vartheta)^2 & \text{if } (**) \\ q_{\text{max}}^2 \sin^4\vartheta & \text{else,} \end{cases} \quad (27)$$

where q_{max} is the maximum of q_1 and q_2 , and (**) is the condition

$$\cos\vartheta < \sqrt{\frac{1 - q_{\text{max}}}{q_{\text{max}}}}. \quad (**)$$

In FIG. 1 we show the gain $P_{\text{opt}} - P_{\text{sep}}$, which of course is always positive or zero. This gain has its absolute maximum of $\frac{1}{4}$ at $q_1 = \frac{1}{2}$ and $\vartheta = \frac{\pi}{3}$. While for fixed angles the maximum gain is always at $q_1 = \frac{1}{2}$, one finds for fixed *a priori* probabilities, that at some regions there are two maxima. The maximum in low values of $\cos\vartheta$ appears, where (**) holds without having (*) satisfied. Also note, that the gain function is asymmetric in $\cos\vartheta$, while it is symmetric in q_1 . In FIG. 2, the gain of the coherent versus the incoherent strategy is illustrated for the parameters $q_1 = \frac{1}{2}$ and $q_1 \rightarrow 1$.

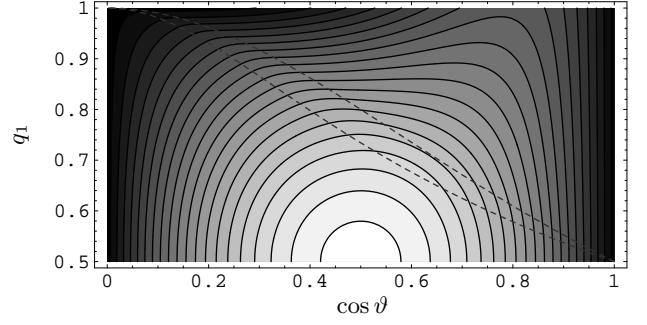
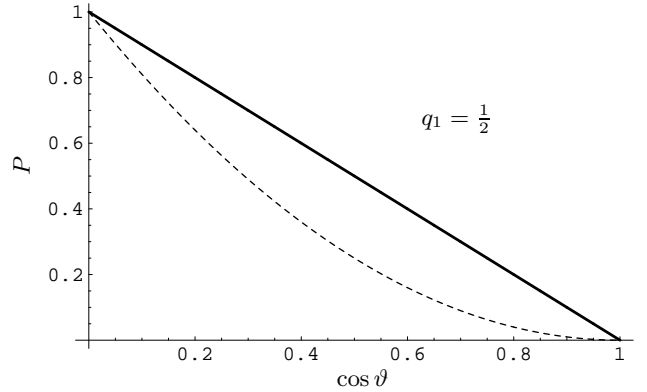


FIG. 1: Contour plot of the gain $P_{\text{opt}} - P_{\text{sep}}$, where higher gain corresponds to brighter shade. White stands for a gain value of 0.25, black for a value of 0.0125, and each contour line corresponds to a step of 0.0125. The dashed lines divide the set of parameters into regions where both (*) and (**) hold (lower left), neither of both condition holds (top right) and (**) holds, but (*) does not (remaining small stripe).



IV. STATE COMPARISON “TWO OUT OF N ”

Next, we investigate the problem of unambiguous state comparison “two out of N ” for pure states. As shown by Chefles *et al.* [2] for equal probabilities and in section V for arbitrary probabilities, this can only work if all N states are linearly independent, thus spanning an

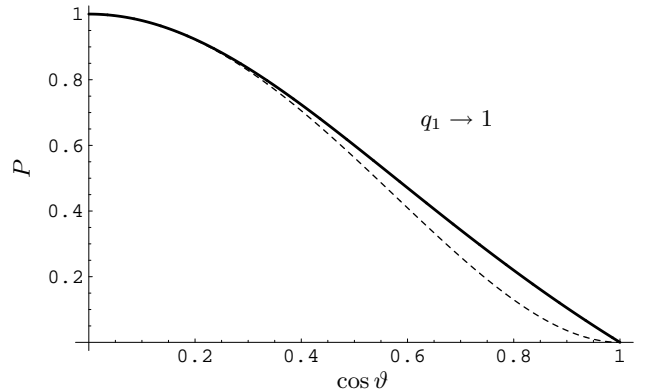


FIG. 2: Rate of success for state comparison “two out of two” with $q_1 = \frac{1}{2}$ (upper graph) and $q_1 \rightarrow 1$ (lower graph). The solid line is the optimal result, and the dashed line corresponds to the best separable measurement.

N -dimensional Hilbert space. Again this unambiguous state comparison is related to the UD of

$$\rho_a = \frac{1}{\eta_a} \sum_i^N q_i^2 |\psi_i \psi_i\rangle \langle \psi_i \psi_i|, \quad (28a)$$

$$\rho_b = \frac{1}{\eta_b} \sum_{i \neq j}^N q_i q_j |\psi_i \psi_j\rangle \langle \psi_i \psi_j|, \quad (28b)$$

having *a priori* probabilities

$$\eta_a = \sum_i q_i^2, \quad \eta_b = \sum_{i \neq j} q_i q_j. \quad (28c)$$

We immediately obtain

$$\mathcal{S}_{\rho_a} = \bigoplus_i \text{span}(|\psi_i \psi_i\rangle), \quad (29a)$$

$$\begin{aligned} \mathcal{S}_{\rho_b} &= \bigoplus_{i \neq j} \text{span}(|\psi_i \psi_j\rangle) \\ &= \bigoplus_{i > j} \text{span}(|\psi_i \psi_j\rangle \pm |\psi_j \psi_i\rangle). \end{aligned} \quad (29b)$$

Due to linear independence $\mathcal{S}_{\rho_a} \cap \mathcal{S}_{\rho_b} = \{0\}$ holds and thus the first reduction step yields $\mathcal{H} = \mathcal{S}_{\rho_a} \oplus \mathcal{S}_{\rho_b}$. Note, that the dimension of \mathcal{S}_{ρ_a} is now in general much smaller than the one of \mathcal{S}_{ρ_b} , because $\dim \mathcal{S}_{\rho_a} = N$ while $\dim \mathcal{S}_{\rho_b} = N^2 - N$. In what follows we show in a constructive way, that the N -dimensional state comparison in general is related to such an UD of mixed states, which cannot be reduced to UD of pure states.

The second reduction step can be performed as follows. The antisymmetric subspace $\mathcal{H}^- = \bigoplus_{i > j} \text{span}(|\psi_i \psi_j\rangle - |\psi_j \psi_i\rangle)$ is part of $\mathcal{K}_a^\cap \equiv \mathcal{K}_{\rho_a} \cap \mathcal{S}_{\rho_b}$, since

$$\mathcal{S}_{\rho_a} \perp \mathcal{H}^- \quad \text{and} \quad \mathcal{S}_{\rho_b} \supset \mathcal{H}^-. \quad (30)$$

Further, \mathcal{S}_{ρ_a} is part of the symmetric subspace $\mathcal{H}^+ = \bigoplus_{i > j} \text{span}(|\psi_i \psi_j\rangle + |\psi_j \psi_i\rangle)$ and thus, due to $\mathcal{H}^- \perp \mathcal{H}^+$, we have the orthogonal decomposition

$$\mathcal{K}_a^\cap = \mathcal{K}_a^{\cap-} \oplus \mathcal{K}_b^{\cap+}, \quad (31)$$

with $\mathcal{K}_a^{\cap-} := \mathcal{H}^-$ and $\mathcal{K}_a^{\cap+} := \mathcal{H}^+ \cap \mathcal{K}_{\rho_a}$. In order to obtain $\mathcal{K}_a^{\cap+}$, let $C_{ij} := \langle \psi_i | \psi_j \rangle$ be the Hermitian overlap matrix and A_{ij} be a lower triangular coefficient matrix. Then $\mathcal{K}_a^{\cap+}$ is given by all vectors $\sum_{i > j} A_{ij} (|\psi_i \psi_j\rangle + |\psi_j \psi_i\rangle)$, which satisfy

$$\begin{aligned} \forall k \quad \langle \psi_k | \sum_{i > j} A_{ij} (|\psi_i \psi_j\rangle + |\psi_j \psi_i\rangle) &= 0 \\ \Leftrightarrow \forall k \quad \sum_{i > j} C_{ki} A_{ij} C_{kj} &= 0 \\ \Leftrightarrow \forall k \quad [CAC^T]_{kk} &= 0. \end{aligned} \quad (32)$$

This set of *linear* equations may eliminate up to N out of $N(N-1)/2$ coefficients A_{ij} , thus

$$\frac{N(N-1)}{2} \geq \dim(\mathcal{K}_a^{\cap+}) \geq \max\{\frac{N(N-3)}{2}, 0\}. \quad (33)$$

The space $\mathcal{K}_b^\cap \equiv \mathcal{K}_{\rho_b} \cap \mathcal{S}_{\rho_a}$ on the other hand is given by all vectors out of \mathcal{S}_{ρ_a} , which are orthogonal to $|\psi_i \psi_j\rangle + |\psi_j \psi_i\rangle$ for all $i > j$. With a diagonal coefficient matrix B this yields

$$\forall i > j \quad [CBC^T]_{ij} = 0. \quad (34)$$

Thus, we have

$$N \geq \dim \mathcal{K}_b^\cap \geq \max\{\frac{N(3-N)}{2}, 0\}. \quad (35)$$

Since the dimension of the reduced Hilbert space is given as $\dim \mathcal{H}' = \dim \mathcal{H} - (\dim \mathcal{K}_a^{\cap-} + \dim \mathcal{K}_a^{\cap+}) - \dim \mathcal{K}_b^\cap$, we finally arrive at the main result of this section,

$$0 \leq \dim \mathcal{H}' \leq \begin{cases} 2 & \text{if } N = 2 \\ 2N & \text{if } N > 2. \end{cases} \quad (36)$$

The case $N = 2$, considered in section III, turns out to play a special role, since here always $\dim \mathcal{K}_b^\cap > 0$ holds, cf. (35). We point out that these bounds are tight. This can be directly verified by considering a system of states with equal overlap, i.e. a system with

$$\cos \vartheta := \langle \psi_i | \psi_j \rangle \in [0, 1[\quad \forall i \neq j. \quad (37)$$

Then for the trivial case (i.e. $\vartheta = \frac{\pi}{2}$) $\dim \mathcal{H}' = 0$ holds, while the upper bound is reached whenever $\vartheta < \frac{\pi}{2}$. Thus state comparison for two out of three states may already lead to a non-trivial UD problem, as illustrated in the following.

Example: “two out of three”

As an example of a case, where state comparison does not reduce to UD of pure states, $N = 3$ is considered. We specialize to the case where the states $|\psi_1\rangle, |\psi_2\rangle$ and $|\psi_3\rangle$ subject to comparison satisfy (37) with $0 < \vartheta < \frac{\pi}{2}$ and assume all *a priori* probabilities to be equal, $q_1 = q_2 = q_3 = \frac{1}{3}$.

The previous discussion of the related UD problem showed, that this related problem can be reduced to a Hilbert space \mathcal{H}' of dimension $\dim \mathcal{H}' = \dim \mathcal{S}_{\rho_a} + \dim \mathcal{S}_{\rho_b} = 3 + 3$. Since $N = 3$ this has the consequence, that $\mathcal{K}_a^{\cap+} = \mathcal{K}_b^\cap = \{0\}$. Thus, \mathcal{H}' exactly consists of the symmetric subspace of $\mathcal{H} \equiv \mathcal{S}_{\rho_a} \oplus \mathcal{S}_{\rho_b}$, i.e. $\mathcal{H}' = \mathcal{H}^+$. However, for the remaining UD problem, no general optimal solution is known and we thus calculate the tightest upper and lower bounds for the rate of success known so far, i.e. the lower bound provided by Rudolph *et al.* [7] and the upper bound shown by Raynal and Lütkenhaus [9]. These bounds together with the rate of success for the separable measurement are shown in FIG. 3. Again, the incoherent measurement is always worse than the measurement used to construct the lower bound. In addition one finds, that for

$$\cos \vartheta \leq \frac{\sqrt{2} - \sqrt{\sqrt{2}}}{2 - \sqrt{2}} \quad (38)$$

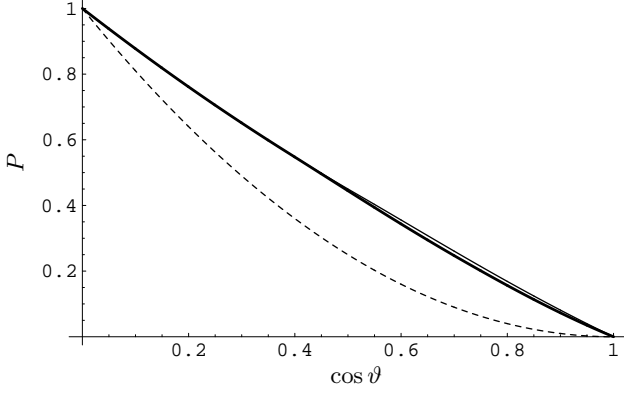


FIG. 3: Bounds for the probability of success for state comparison “two out of three”, with equal *a priori* probabilities and relative angles. The solid lines are an upper [9] and a lower bound [7], while the dashed line corresponds to the separable measurement.

(i.e. $\vartheta/\pi \gtrsim 0.375$) the lower and upper bound coincide, revealing the *optimal* solution of UD of ρ_a and ρ_b in that region to be

$$P_{\text{opt}} = 1 - \frac{\sqrt{8}}{9}(4 \cos \vartheta - \cos^2 \vartheta). \quad (39)$$

One can also show, that in this region the optimal measurement satisfies the defining property (2) and thus also solves the problem of optimal state comparison.

V. MIXED STATE COMPARISON

In this section we investigate, in what situations a measurement can exist, which satisfies the defining property (2). We have the following

Proposition 1 *Unambiguous state comparison “C out of N” for a set of mixed states $\{\pi_1, \dots, \pi_N\}$ with arbitrary, but non-vanishing a priori probabilities, can be realized iff $\forall i$*

$$\mathcal{S}_{\pi_i} \not\subseteq \sum_{k \neq i} \mathcal{S}_{\pi_k}. \quad (40)$$

Proof. For the *if* part it is enough to show, that there is a POVM, given by $\{\tilde{F}_1, \dots, \tilde{F}_N, \tilde{F}_?\}$, such that

$$\text{tr}(\tilde{F}_i \pi_j) > 0 \quad \Leftrightarrow \quad i = j. \quad (41)$$

In order to construct such an POVM, denote by P_i the projector onto the ortho-complement of $\sum_{k \neq i} \mathcal{S}_{\pi_k}$. Then from equation (40) it follows, that there is at least one vector $|\varphi\rangle \in \mathcal{S}_{\pi_i}$, such that $|\phi_i\rangle := P_i|\varphi\rangle$ satisfies $\langle\phi_i|\phi_i\rangle = 1$. These vectors $|\phi_i\rangle$ by construction satisfy $\langle\phi_i|\pi_i|\phi_i\rangle > 0$ for each i , while $\langle\phi_i|\pi_j|\phi_i\rangle = 0$ for all $j \neq i$. The choice $\tilde{F}_i = \frac{1}{N}|\phi_i\rangle\langle\phi_i|$ satisfies (41) and further has $\tilde{F}_? = \mathbb{1} - \sum_i \tilde{F}_i \geq 0$. Indeed, for any $|\psi\rangle$ out of

the complete Hilbert space,

$$\langle\psi|\tilde{F}_?|\psi\rangle = \langle\psi|\psi\rangle - \frac{1}{N} \sum_i |\langle\phi_i|\psi\rangle|^2 \geq 0 \quad (42)$$

holds by virtue of the Cauchy-Schwarz inequality.

For the *only if* part we use, that any unambiguous state comparison measurement solves (not necessarily in an optimal way) the related unambiguous state discrimination problem. However, assuming that for some i :

$$\mathcal{S}_{\pi_i} \subset \sum_{k \neq i} \mathcal{S}_{\pi_k}, \quad (43)$$

we show, that no UD measurement can satisfy $\text{tr}(F_a \pi_i^{\otimes C}) > 0$, thus being a contradiction to (2a).

In order to show this contradiction, note, that for positive operators A and B ,

$$\mathcal{S}_{A+B} = \mathcal{S}_A + \mathcal{S}_B, \quad (44a)$$

$$\mathcal{S}_{A \otimes B} = \mathcal{S}_A \otimes \mathcal{S}_B. \quad (44b)$$

Further we use a Lemma, shown by Raynal, Lütkenhaus and van Enk in [4], which states, that $\text{tr}(AB) = 0$, iff $\mathcal{S}_A \perp \mathcal{S}_B$. Now, assuming (43), it follows that

$$\mathcal{S}_{\pi_i^{\otimes C}} = \mathcal{S}_{\pi_i}^{\otimes C} \subset \sum_{k \neq i} \mathcal{S}_{\pi_k} \otimes \mathcal{S}_{\pi_i}^{\otimes (C-1)} \subset \mathcal{S}_{\rho_b}. \quad (45)$$

However, by the Lemma of [4], the requirement $\text{tr}(F_a \rho_b) = 0$ (cf. (5)) is equivalent to $\mathcal{S}_{F_a} \perp \mathcal{S}_{\rho_b}$. This implies $\mathcal{S}_{F_a} \perp \mathcal{S}_{\pi_i^{\otimes C}}$ or equivalently $\text{tr}(F_a \pi_i^{\otimes C}) = 0$ and completes the proof. ■

For the comparison of qubits this proposition implies that unambiguous comparison “C out of N” can only be realized for $N = 2$ and *pure states*. For unambiguous state comparison “C out of N” of pure states in any dimension, Proposition 1 reduces to the result of Chefles *et al.* [2]. They found that state comparison can only be realized for linearly independent states. Another direct consequence from Proposition 1 is the fact that density matrices which contain a proportion of the identity (e.g. by being sent through a depolarising channel, or by adding white noise in an experiment) can never be compared unambiguously.

VI. CONCLUSIONS

We have addressed the question of unambiguous state comparison with general *a priori* probabilities. Our method consists of reducing the corresponding problem of unambiguous mixed state discrimination to a non-trivial subspace [4]. We analytically solve the case for comparing two states drawn from a set of two states, finding the optimal POVMs and the optimal rate of success. There is a considerable gain of the optimal coherent strategy over the best incoherent strategy. While this case reduces to the discrimination between two pure states, the

comparison of two states drawn from a set of three states is shown to lead to a non-trivial mixed state discrimination task. So far, the optimal solution is only found for certain parameter ranges.

The more general task of comparing two states from a set of N states is exceedingly difficult. No general solution to this problem exists. Here, we have presented an upper bound for the dimension of the reduced Hilbert space. This bound is shown to be reached for states with equal overlap. We have also provided a necessary and sufficient condition for unambiguous comparison of mixed states to be possible.

Note added: While completing this manuscript, we learned about related work by Herzog and Bergou [11], who found the same expression as equation (22) for optimal unambiguous state comparison of two states drawn from a set of two states.

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APPENDIX A: OPTIMAL SEPARABLE MEASUREMENT “TWO OUT OF TWO”

This appendix is dedicated to show that with the naïve measurement given in equation (26), indeed the optimal separable solution was found. That is, the optimal *separable* unambiguous state comparison measurement for two states drawn from a set of two pure states $\{|\psi_1\rangle, |\psi_2\rangle\}$ is solved in an optimal way by performing optimal unambiguous state discrimination in each subsystem.

A general element of a separable POVM $\{F_x\}$ is of the form

$$F_x = \sum_{i,j} c_{x,ij} F_{x,i}^{(1)} \otimes F_{x,j}^{(2)}, \quad (\text{A1})$$

where the non-negative coefficients $c_{x,ij}$ account for the relative contribution of each of the terms containing the positive local POVM elements $F_{x,i}^{(k)}$.

First we show, that in our case no measurement outcome of either subsystem can be used to adapt the measurement of the other. Consider without loss of generality, that a measurement first takes place in subsystem 1 and yields with probability $p_{x,i}^{(1)}$ the outcome (x, i) . This measurement is applied to the global state $\rho := \eta_a \rho_a + \eta_b \rho_b = (q_1 |\psi_1\rangle\langle\psi_1| + q_2 |\psi_2\rangle\langle\psi_2|)^{\otimes 2}$ and yields in subsystem 2

$$\text{tr}_1((F_{x,i}^{(1)} \otimes \mathbb{1})\rho) = p_{x,i}^{(1)}(q_1 |\psi_1\rangle\langle\psi_1| + q_2 |\psi_2\rangle\langle\psi_2|), \quad (\text{A2})$$

which is, up to the factor $p_{x,i}^{(1)}$ independent of the outcome (x, i) . Thus the local measurements can be optimized in each subsystem separately, and one is free to choose the same (optimal) measurement in both systems due to the symmetry of ρ_a and ρ_b . Therefore we can drop the upper label (k) on the local measurement elements in the following.

Furthermore one is forced to choose these measurements to be UD measurements. Indeed, $\text{tr}(F_a \rho_b) = \text{tr}(F_b \rho_a) = 0$, only if for each $x \in \{a, b\}$ and for all l , either $\text{tr}(F_{x,l} |\psi_1\rangle\langle\psi_1|) = 0$ or $\text{tr}(F_{x,l} |\psi_2\rangle\langle\psi_2|) = 0$. We prove this statement by contradiction: Suppose, that at least one term $(c_{a,ij} F_{a,i} \otimes F_{a,j})$ of F_a contains at least one local POVM element $F_{a,m}$ (where $m \in \{i, j\}$), having a non-vanishing expectation value for both states, i.e.

$$\langle\psi_1|F_{a,m}|\psi_1\rangle > 0 \quad \text{and} \quad \langle\psi_2|F_{a,m}|\psi_2\rangle > 0. \quad (\text{A3})$$

It follows that

$$\text{tr}((c_{x,ij} F_{a,i} \otimes F_{a,j})\rho_a) > 0 \quad (\text{A4})$$

and

$$\text{tr}((c_{x,ij} F_{a,i} \otimes F_{a,j})\rho_b) > 0, \quad (\text{A5})$$

which is in contradiction to $\text{tr}(F_a \rho_b) = 0$. An analogous argument holds F_b .

Without loosing any information, a UD measurement can always be reduced to have the measurement elements $\{F_1, F_2, F_?\}$, with $\langle\psi_2|F_1|\psi_2\rangle = \langle\psi_1|F_2|\psi_1\rangle = 0$. In order to make this a valid choice for the local measurements of unambiguous state comparison, in addition the conditions (2) have to be satisfied, i.e.

$$\alpha := \langle\psi_1|F_1|\psi_1\rangle > 0 \quad \text{and} \quad \beta := \langle\psi_2|F_2|\psi_2\rangle > 0. \quad (\text{A6})$$

From the consideration above, we find that F_a and F_b are of the form

$$F_a = F_1 \otimes F_1 + F_2 \otimes F_2, \quad (\text{A7})$$

$$F_b = F_1 \otimes F_2 + F_2 \otimes F_1. \quad (\text{A8})$$

The optimal separable state comparison corresponds to $F_1 = \tilde{F}_1$ and $F_2 = \tilde{F}_2$ as above defined (26). Thus we have shown, that in this case the optimal separable unambiguous state comparison strategy is indeed given by consecutive optimal UD measurements.

Let us mention that for the optimal UD measurement the conditions $\alpha > 0$ and $\beta > 0$ do not always hold: in those situations, where condition (**) is not satisfied, $\alpha = 0$ or $\beta = 0$. But changing α and β (under the constraint $\mathbb{1} - F_1 - F_2 \geq 0$) infinitesimally, affects the probability of success only infinitesimally. In this limit, we consider the optimal unambiguous state discrimination measurement as a valid choice for F_1 and F_2 .

We conjecture that also in the more general scenario of unambiguous state comparison of “C out of N” states, the best separable measurement is given by performing unambiguous state discrimination in each subsystem. However, the proof by contradiction given above for “two out

of two” cannot be generalized in a straightforward way for the operator F_b . We leave the generalization as an

open question for future work.

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